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Some Properties and Generalizations of Multivariate Eyraud–Gumbel-Morgenstern Distributions*

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The admissible values of the coefficient in a bivariate Eyraud–Gumbel–Morgenstern (EGM) distribution are found. For multivariate EGM distributions necessary and sufficient conditions are given for its coefficients, and its conditional distributions are found and shown to belong to a family of distributions further extending the multivariate EGM family.

1. INTRODUCTION

Eyraud [1] and independently Morgenstern [5] and Gumbel [2] introduced a family of bivarate distributions $H(x_1, x_2)$, with marginals $F_1(x_1)$ and $F_2(x_2)$, of the form

$$H(x_1, x_2) = F_1(x_1) F_2(x_2)\{1 + \alpha[1 - F_1(x_1)][1 - F_2(x_2)]\} \tag{1.1}$$

where α is a real constant. In Section 2 we find all values of α for which H as defined by (1.1) is a bivariate distribution, assuming of course that F_1 and F_2 are univariate distributions.

Johnson and Kotz [3] introduced a multivariate Eyraud–Gumbel–Morgenstern (EGM) family of distributions $H(x_1, x_2, \ldots, x_n)$, with marginals $F_1(x_1), F_2(x_2), \ldots, F_n(x_n)$, by

$$\begin{aligned} H(x_1, x_2, \ldots, x_n) &= F_1(x_1) F_2(x_2) \cdots F_n(x_n) \Big\{ 1 + \sum_{1 \leqslant j_1 < j_2 \leqslant n} \alpha_{j_1 j_2}[1 - F_{j_1}(x_{j_1})][1 - F_{j_2}(x_{j_2})] \\ &\quad + \cdots + \alpha_{12\cdots n}[1 - F_1(x_1)][1 - F_2(x_2)] \cdots [1 - F_n(x_n)] \Big\} \end{aligned} \tag{1.2i}$$

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where the coefficients α are real constants. Omitting the variables we can write (1.2i) in the more compact form

$$H = F_1 \cdots F_n \left\{ 1 + \sum_{p=2}^{n} \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} \alpha_{j_1 \cdots j_p} (1 - F_{j_1}) \cdots (1 - F_{j_p}) \right\}. \quad (1.2\text{ii})$$

In Section 2 we also give necessary and sufficient conditions on the coefficients α so that (1.2) defines an n-dimensional distribution, assuming again that $F_1, ..., F_n$ are univariate distributions.

In Section 4 we introduce a family of distributions closely related to the multivariate EGM family, by inserting within the brackets on the right-hand side of (1.2) the term $\sum_{j=1}^{n} \alpha_j [1 - F_j(x_j)]$, and we show in Section 3 that if $X_1, ..., X_n$ have an n-dimensional EGM distribution, then the conditional distribution of $X_1, ..., X_k$ given $X_{k+1} = x_{k+1}, ..., X_n = x_n$ $(k = 1, ..., n-1)$ belongs to this family.

2. The Values of the Coefficients α

For a distribution function $F_i(x)$ let E_i be the set of all values of $F_i(x)$ with the exception of 0 and 1, i.e., E_i is the subset of (0, 1) defined by

$$E_i = \{F_i(x), -\infty < x < +\infty\} - \{0, 1\},$$

and let

$$m_i = \text{g.l.b. } E_i \ (= \inf E_i), \qquad M_i = \text{l.u.b. } E_i \ (= \sup E_i).$$

Then clearly $0 \leqslant m_i \leqslant M_i \leqslant 1$ and we have the following

THEOREM 1. *H defined by* (1.1) *is a bivariate distribution if and only if*

$$\alpha_{\min} = -\min \left\{ \frac{1}{M_1 M_2}, \frac{1}{(1 - m_1)(1 - m_2)} \right\}$$
$$\leqslant \alpha \leqslant \min \left\{ \frac{1}{M_1(1 - m_2)}, \frac{1}{(1 - m_1) M_2} \right\} = \alpha_{\max}.$$

Proof. $H(x_1, x_2)$ is a bivariate distribution if and only if for all $x_1 < x_1'$ and $x_2 < x_2'$ we have $\Delta_{x_1'} \Delta_{x_2'} H(x_1, x_2) \geqslant 0$, where $\Delta_{x'} V(x) = V(x') - V(x)$. But by (1.1) it follows that

$$\begin{aligned} \Delta_{x_1'} \Delta_{x_2'} H(x_1, x_2) &= \Delta_{x_1'} F_1(x_1)\, \Delta_{x_2'} F_2(x_2) \\ &\quad + \alpha \Delta_{x_1'} \{F_1(x_1)[1 - F_1(x_1)]\}\, \Delta_{x_2'} \{F_2(x_2)[1 - F_2(x_2)]\} \\ &= \{1 + \alpha A_1(x_1, x_1')\, A_2(x_2, x_2')\}\, \Delta_{x_1'} F_1(x_1)\, \Delta_{x_2'} F_2(x_2) \end{aligned}$$

where $A_i(x, x') = 1 - F_i(x) - F_i(x')$, $i = 1, 2$. Hence H is a bivariate distribution if and only if

$$1 + \alpha A_1(x_1, x_1') A_2(x_2, x_2') \geqslant 0 \qquad \text{for all } \Delta_{x_i'} F_i(x_i) \geqslant 0, \quad i = 1, 2. \quad (2.1)$$

From the definitions of m and M, we clearly have

$$\begin{aligned} &\text{g.l.b. } \{F(x) + F(x') \colon \Delta_{x'} F(x) > 0\} = m, \\ &\text{l.u.b. } \{F(x) + F(x') \colon \Delta_{x'} F(x) > 0\} = 1 + M. \end{aligned}$$

It follows, omitting for simplicity the x's, that

$$\text{g.l.b. } \{A \colon \Delta F > 0\} = -M, \qquad \text{l.u.b.} \{A \colon \Delta F > 0\} = 1 - m.$$

Hence

$$\begin{aligned} &\text{g.l.b. } \{A_1 A_2 \colon \Delta F_1 > 0, \Delta F_2 > 0\} = -\max\{M_1(1 - m_2), (1 - m_1) M_2\}, \\ &\text{l.u.b. } \{A_1 A_2 \colon \Delta F_1 > 0, \Delta F_2 > 0\} = \max\{M_1 M_2, (1 - m_1)(1 - m_2)\}, \end{aligned}$$

and the bounds for α given in the theorem follow from (2.1). ∎

Notice that $\alpha_{\min} \leqslant -1$ and $1 \leqslant \alpha_{\max}$, and that in fact

$$\begin{aligned} &\alpha_{\min} = -1 \qquad \text{if and only if } M_1 = M_2 = 1 \text{ or } m_1 = m_2 = 0, \\ &\alpha_{\max} = 1 \qquad \text{if and only if } (M_1 = 1 \text{ and } m_2 = 0) \text{ or } (m_1 = 0 \text{ and } M_2 = 1). \end{aligned}$$

When the marginals are identical, $F_1 = F_2 = F$, then

$$\alpha_{\min} = -\frac{1}{\{\max(M, 1 - m)\}^2} \leqslant \alpha \leqslant \frac{1}{M(1 - m)} = \alpha_{\max}.$$

As an example, when $F(x) = 0$ for $x < 0$, $= p$ for $0 \leqslant x < 1$, $= 1$ for $1 \leqslant x$, with $0 < p < 1$, then $M = m = p$ and the admissible values of α are

$$-\frac{1}{\{\max(p, 1 - p)\}^2} \leqslant \alpha \leqslant \frac{1}{p(1 - p)}.$$

This example is considered by Johnson and Kotz [3].

If F has a density, then $m = 0$ and $M = 1$. Thus if both F_1 and F_2 have densities, then the admissible values of α are $-1 \leqslant \alpha \leqslant 1$, a result obtained by Johnson and Kotz [3]. It is clear, however, that we may have $m = 0$ and $M = 1$ even when F is not absolutely continuous. For instance if F is a discrete distribution with (positive) jumps at the integers (or at x_n with $\inf x_n = -\infty$ and $\sup x_n = +\infty$), then $m = 0$ and $M = 1$.

The same method shows that $H(x_1, \ldots, x_n)$ defined by (2) is an n-dimensional distribution if and only if

$$1 + \sum_{1 \leqslant j_1 < j_2 \leqslant n} \alpha_{j_1 j_2} A_{j_1}(x_{j_1}, x'_{j_1}) A_{j_2}(x_{j_2}, x'_{j_2}) + \cdots$$
$$+ \alpha_{1 \ldots n} A_1(x_1, x_1') \cdots A_n(x_n, x_n') \geqslant 0$$

whenever $\Delta_{x_i'} F(x_i) > 0$, $i = 1, \ldots, n$. Since $-M_i \leqslant A_i(x_i, x_i') \leqslant 1 - m_i$ whenever $\Delta_{x_i'} F({}_i x_i) > 0$, it follows that H is a distribution if and only if the following 2^n inequalities are satisfied by the α's,

$$1 + \sum_{1 \leqslant j_1 < j_2 \leqslant n} \epsilon_{j_1} \epsilon_{j_2} \alpha_{j_1 j_2} + \cdots + \epsilon_1 \cdots \epsilon_n \alpha_{1 \ldots n} \geqslant 0$$

where for each $i = 1, \ldots, n$, $\epsilon_i = -M_i$ or $1 - m_i$. These conditions were obtained by Johnson and Kotz [3] under the assumption that all marginals F_i have densities, in which case $\epsilon_i = \pm 1$.

If $H(x_1, \ldots, x_n)$ has the following (simplest possible symmetric) form $H = F_1 \cdots F_n\{1 + \alpha(1 - F_1) \cdots (1 - F_n)\}$, then the admissible values of α are

$$-\frac{1}{\max\{\gamma_1 \cdots \gamma_n\}} \leqslant \alpha \leqslant \frac{1}{\max\{\delta_1 \cdots \delta_n\}}$$

where the maxima are taken over all products with each $\gamma_i = M_i$ or $1 - m_i$ and an even number of γ_i's equal to M_i, and each $\delta_i = M_i$ or $1 - m_i$ and an odd number of δ_i's equal to M_i.

3. The Conditional Distributions

In this section we compute the (regular) conditional distribution of $X_1, \ldots, X_k$ given $X_{k+1} = x_{k+1}, \ldots, (X_n = x_n)(k = 1, \ldots, n-1)$ when $X_1, \ldots, X_n$ have a multivariate EGM distribution.

Let the random variables $X_1, \ldots, X_n$ have an EGM distribution given by (1.2). For $j_1 < \cdots < j_m$ we denote by $H_{j_1 \cdots j_m}$ the distribution of $X_{j_1}, \ldots, X_{j_m}$ (which is also EGM). For convenience we will use the same symbol for a distribution as for its corresponding (Lebesgue–Stieltjes) probability measure. Also if the finite (possibly signed) measure λ is absolutely continuous with respect to the finite measure ν, $\lambda \ll \nu$, we will denote by $[d\lambda/d\nu]$ the corresponding Radon–Nikodym derivative. We need the following simple result.

Lemma. *If F is a univariate distribution, then $F(1 - F) \ll F$ and*

$$\left[\frac{d[F(1 - F)]}{dF}\right](x) = 1 - F(x) - F(x-) = B(x).$$

Proof. For each u we have

$$\int_{(-\infty,u]} F(x)\,dF(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{(-\infty,u]}(x)\,\chi_{(-\infty,x]}(y)\,dF(y)\,dF(x)$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \chi_{(-\infty,u]}(x)\,\chi_{[y,\infty)}(x)\,dF(x)\right) dF(y)$$

$$= \int_{(-\infty,u]} [F(u) - F(y-)]\,dF(y)$$

$$= F^2(u) - \int_{(-\infty,u]} F(y-)\,dF(y)$$

and thus

$$\int_{(-\infty,u]} [1 - F(x) - F(x-)]\,dF(x) = F(u) - F^2(u) = F(u)[1 - F(u)]$$

$$= \int_{(-\infty,u]} d[F(1-F)]$$

which concludes the proof. ∎

It then follows from (1.2) and the lemma that $H_{1\cdots n} \ll F_1 \times \cdots \times F_n$ with

$$\left[\frac{dH_{1\cdots n}}{dF_1 \cdots dF_n}\right] = 1 + \sum_{p=2}^{n} \sum_{1\leqslant j_1<\cdots<j_p\leqslant n} \alpha_{j_1\cdots j_p} B_{j_1} \cdots B_{j_p}. \tag{3.1}$$

For $j_1 < \cdots < j_m$ we will denote $[dH_{j_1\cdots j_m}/dF_{j_1} \cdots dF_{j_m}]$ by $d_{j_1\cdots j_m}$, and we have a similar expression for it.

THEOREM 2. *With the above assumptions and notation the function* $H_{1\cdots k|k+1\cdots n}(x_1, \ldots, x_k \mid x_{k+1}, \ldots, x_n)$ *defined by*

$$\frac{F_1(x_1)\cdots F_k(x_k)}{d_{k+1\cdots n}(x_{k+1}, \ldots, x_n)} \left\{1 + \sum_{p=2}^{n} \sum_{1\leqslant j_1<\cdots<j_p\leqslant n} \alpha_{j_1\cdots j_p} C_{j_1}^{(k)}(x_{j_1}) \cdots C_{j_p}^{(k)}(x_{j_p})\right\} \tag{3.2}$$

when $d_{k+1\cdots n}(x_{k+1}, \ldots, x_n) > 0$ *(and otherwise by, say,* $F_1(x_1)\cdots F_k(x_k)$*) and where* $C_j^{(k)}(x) = 1 - F_j(x)$ *for* $1 \leqslant j \leqslant k$ *and* $1 - F_j(x) - F_j(x-) = B_j(x)$ *for* $k < j \leqslant n$ *is a regular conditional distribution of* $X_1, \ldots, X_k$ *given* $X_{k+1} = x_{k+1}, \ldots, X_n = x_n$.

Proof. A standard argument (which is therefore omitted here) shows that when $d_{k+1\cdots n}(x_{k+1}, \ldots, x_n) > 0$, a regular conditional distribution of $X_1, \ldots, X_k$ given $X_{k+1} = x_{k+1}, \ldots, X_n = x_n$ equals

$$\frac{\int_{(-\infty,x_1]} \cdots \int_{(-\infty,x_k]} d_{1\cdots n}(u_1, \ldots, u_k, x_{k+1}, \ldots, x_n)\,dF_1(u_1)\cdots dF_k(u_k)}{d_{k+1\cdots n}(x_{k+1}, \ldots, x_n)}.$$

Equation (3.2) then follows by using (3.1) and the lemma, which shows that

$$\int_{(-\infty,x]} B(u)\, dF(u) = F(x)[1 - F(x)]. \quad \blacksquare$$

Since values $(x_{k+1}, \ldots, x_n)$ with $d_{k+1\cdots n}(x_{k+1}, \ldots, x_n) = 0$ are taken by $(X_{k+1}, \ldots, X_n)$ with probability zero, the expression of $H_{1\cdots k|k+1\cdots n}$ given by (3.2) is the one of interest and it is only for such $(x_{k+1}, \ldots, x_n)$'s that expressions will be written out in the sequel.

By rearranging terms in (3.2) and noting that, as in (3.1),

$$d_{k+1\cdots n} = 1 + \sum_{p=k+1}^{n} \sum_{k+1 \leqslant j_1 < \cdots < j_p \leqslant n} \alpha_{j_1\cdots j_p} B_{j_1} \cdots B_{j_p}$$

we can write

$$H_{1\cdots k|k+1\cdots n} = F_1 \cdots F_k \left\{ 1 + \sum_{p=1}^{k} \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant k} \beta^{(k)}_{j_1\cdots j_p} (1 - F_{j_1}) \cdots (1 - F_{j_p}) \right\} \tag{3.3}$$

where

$$d_{k+1\cdots n} \beta^{(k)}_{j_1} = D^{(k)}_{j_1}, \tag{3.4i}$$

$$d_{k+1\cdots n} \beta^{(k)}_{j_1\cdots j_p} = \alpha_{j_1\cdots j_p} + D^{(k)}_{j_1\cdots j_p}, \qquad 1 < p \leqslant k, \tag{3.4ii}$$

$$D^{(k)}_{j_1\cdots j_p} = \sum_{q=1}^{n-k} \sum_{k+1 \leqslant i_1 < \cdots < i_q \leqslant n} \alpha_{j_1\cdots j_p i_1 \cdots i_q} B_{i_1} \cdots B_{i_q}, \tag{3.5}$$

and all β's are clearly functions of $x_{k+1}, \ldots, x_n$ only. Note that $H_{1\cdots k/k+1\cdots n}$ is a k-variate EGM distribution only if $\beta_j^{(k)} = D_j^{(k)} = 0$ for all $j = 1, \ldots, k$.

When $k = n - 1$, or when $X_{k+1}, \ldots, X_n$ are independent, we have $d_{k+1\cdots n} = 1$ and in view of (3.4), (3.3) can be written as

$$H_{1\cdots k|k+1\cdots n} = H_{1\cdots k} + F_1 \cdots F_k \sum_{p=1}^{k} \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant k} D^{(k)}_{j_1\cdots j_p} (1 - F_{j_1}) \cdots (1 - F_{j_p}).$$

Thus whenever $f(X_1, \ldots, X_k)$ has finite expectation, we have

$$\begin{aligned} &\mathscr{E}[f(X_1, \ldots, X_k) \mid X_{k+1}, \ldots, X_n] \\ &\quad = \mathscr{E}[f(X_1, \ldots, X_k)] + \sum_{p=1}^{k} \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant k} a^{(k)}_{j_1\cdots j_p} D^{(k)}_{j_1\cdots j_p}(X_{k+1}, \ldots, X_n) \end{aligned}$$

where

$$\begin{aligned} a^{(k)}_{j_1\cdots j_p} &= \int_{R^k} f(x_1, \ldots, x_k)\, B_{j_1}(x_1) \cdots B_{j_p}(x_p)\, dF_1(x_1) \cdots dF_k(x_k) \\ &= \mathscr{E}\left[\frac{f(X_1, \ldots, X_k)\, B_{j_1}(X_1) \cdots B_{j_p}(X_{j_p})}{d_{1\cdots k}(X_1, \ldots, X_k)} \right]. \end{aligned}$$

In particular we find that if the means $\mu_i = \mathscr{E}(X_i)$ exist, then $(i = 1,..., n-1)$

$$\begin{aligned}\mathscr{E}(X_i \mid X_n) &= \mathscr{E}(X_i) - c_i \alpha_{in} B_n(X_n),\\ \mathscr{E}\{(X_1 - \mu_1) \cdots (X_{n-1} - \mu_{n-1}) \mid X_n\} &= \mathscr{E}\{(X_1 - \mu_1) \cdots (X_{n-1} - \mu_{n-1})\}\\ &\quad + (-1)^{n-1} c_1 \cdots c_{n-1} \alpha_{1\cdots n} B_n(X_n),\end{aligned}$$

where the c's are defined by (4.3) in Section 4.

Also, for $k = 1$, we have

$$H_{1|2\cdots n}(x_1 \mid x_2, ..., x_n) = F_1(x_1)\{1 + \beta(x_2, ..., x_n)[1 - F_1(x_1)]\}$$

where β is given by (3.4) and (3.5), and thus

$$\mathscr{E}[f(X_1) \mid X_2, ..., X_n] = \mathscr{E}[f(X_1)] + \beta(X_2, ..., X_n)\, \mathscr{E}[f(X_1)\, B_1(X_1)].$$

4. A Generalization of the EGM Family

In this section we introduce a family of distributions which constitutes a natural generalization of the multivariate EGM family. This family has some interest on its own but its raison d'être here is the fact that it contains all conditional distributions of the multivariate EGM family.

Let $F_1(x_1),..., F_n(x_n)$ be univariate distributions and consider the family $\mathscr{M}$ of multivariate distributions $H(x_1, ..., x_n)$ defined by

$$\begin{aligned}H(x_1, ..., x_n) = F_1(x_1) \cdots F_n(x_n) \Big\{ 1 &+ \sum_{j=1}^{n} \beta_j [1 - F_j(x_j)]\\ &+ \sum_{1 \leqslant j_1 < j_2 \leqslant n} \beta_{j_1 j_2} [1 - F_{j_1}(x_{j_1})][1 - F_{j_2}(x_{j_2})] + \cdots\\ &+ \beta_{1\cdots n} [1 - F_1(x_1)] \cdots [1 - F_n(x_n)] \Big\}\end{aligned} \tag{4.1}$$

where the coefficients β are real constants. Clearly this family contains all EGM distributions, for which $\beta_j = 0$, $j = 1,..., n$, as well as all conditional distributions of an EGM distribution (cf. (3.3)).

Notice that (4.1) differs from (1.2) only by the introduction of the first-order terms: $\sum_{j=1}^{n} \beta_j [1 - F_j(x_j)]$. As a result the marginal distributions $H_i(x_i)$ of H defined by (4.1) are now given by

$$H_i(x_i) = F_i(x_i)\{1 + \beta_i [1 - F_i(x_i)]\} \tag{4.2}$$

and are not equal to the original set of univariate distributions F_i, unless all β_i equal zero. Equation (4.2) is not a restriction on the marginals H_i, since every distribution $H(x)$ can always be written in the form $H(x) = F(x)\{1 - \beta[1 - F(x)]\}$, for some distribution $F(x)$ and some real number β. In fact given any distribution $H(x)$ and any real β there is a distribution $F(x)$ satisfying this equality.

We now state some properties of the family $\mathscr{M}$, their proof being either standard or similar to earlier proofs.

All marginal and conditional distributions of an element in $\mathscr{M}$ belongs to $\mathscr{M}$.

The necessary and sufficient conditions on the coefficients β for H defined by (4.1) to be an n-dimensional distribution are the following

$$-(1/(1 - m_i)) \leqslant \beta_i \leqslant 1/M_i, \qquad i = 1,..., n,$$

$$1 + \sum_{j=1}^{n} \epsilon_j\beta_j + \sum_{1\leqslant j_1<j_2\leqslant n} \epsilon_{j_1}\epsilon_{j_2}\beta_{j_1j_2} + \cdots + \epsilon_1 \cdots \epsilon_n\beta_{1\cdots n} \geqslant 0,$$

where for each $i = 1,..., n$, $\epsilon_i = -M_i$ or $1 - m_i$.

Let $X_1 ,..., X_n$ be random variables with joint distribution in $\mathscr{M}$. Then $X_1 ,..., X_n$ are independent if and only if for all $k = 2,..., n$ and $j_1 < \cdots < j_k$ we have

$$\beta_{j_1\cdots j_k} = \beta_{j_1} \cdots \beta_{j_k} .$$

The full meaning of the coefficients β is given by the following relationship

$$\mathscr{E}\{(X_{j_1} - \mu'_{j_1}) \cdots (X_{j_k} - \mu'_{j_k})\} = (-1)^k c_{j_1} \cdots c_{j_k}\beta_{j_1\cdots j_k}$$

where, omitting the subscript,

$$c = \int_{-\infty}^{\infty} F(x)\{1 - F(x)\}\,dx = -\int_{-\infty}^{\infty} xd\{F(x)[1 - F(x)]\}$$
$$= -\int_{-\infty}^{\infty} x\{1 - F(x) - F(x-)\}\,dF(x), \qquad (4.3)$$

$$\mu_i' = \int_{-\infty}^{\infty} x_i dF_i(x_i),$$

and where the following assumption is needed to guarantee that all integrals and expectations are finite: $\int_{-\infty}^{\infty} | x_i | \, dF_i(x_i) < \infty$, $i = 1,..., n$. The functionals c of F were first introduced by Johnson and Kotz [4] in evaluating the regression $\mathscr{E}(X_1 \mid X_2)$ when X_1, X_2 have a bivariate EGM distribution. Also

$$\mu_i = \mathscr{E}(X_i) = \mu_i' - c_i\beta_i$$

and for the EGM family we have $\mu_i = \mu_i'$ and

$$\mathscr{E}\{(X_{j_1} - \mu_{j_1}) \cdots (X_{j_k} - \mu_{j_k})\} = (-1)^k c_{j_1} \cdots c_{j_k}\alpha_{j_1\cdots j_k} .$$

Finally

$$\mathrm{Cov}(X_{j_1}, X_{j_2}) = c_{j_1} c_{j_2} (\beta_{j_1 j_2} - \beta_{j_1} \beta_{j_2})$$

shows that if X_{j_1} and X_{j_2} are uncorrelated then they are independent.

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